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1985 J. Phys. A: Math. Gen. 18 2795

(http://iopscience.iop.org/0305-4470/18/14/029)

J. Phys. A: Math. Gen. 18 (1985) 2795-2806. Printed in Great Britain

Site decorations and critical exponents in percolation

G Ord† and S G Whittington‡

† Cornell University, Baker Laboratory, Ithaca, NY 14853, USA

‡ Department of Chemistry, University of Toronto, Toronto, Ontario, Canada, M5S 1A1

Received 10 December 1984

Abstract. We introduce and discuss a family of site decorations which change local lattice features, such as coordination number and planarity, but which may be shown explicitly to preserve the usual percolation exponents even in a continuum limit. The continuum limit is shown to be equivalent to cell percolation where sites are randomly and independently distributed in space, but where connectivity is determined by a cell structure. We further verify conjectured amplitude relationships within lattice families without recourse to scaling arguments.

1. Introduction

From the study of thermal critical phenomena there is considerable evidence that critical exponents depend on the dimension of the problem but not on the details of lattice structures within that dimension. Through contact with the Potts model (Kasteleyn and Fortuin 1969) and from direct numerical evidence (Gaunt and Sykes 1983, and references therein) it is believed that exponents in percolation are also dependent only on the dimension of the lattice. Unfortunately rigorous calculations of percolation exponents have not yet been carried out for any pair of common lattices, so that such universality has not yet been proved in any dimension. However, in a previous paper (Ord, Whittington and Wilker (1984) abbreviated OWW) we showed that bond problems could be decorated in a non-trivial way so as to preserve the exponent β while changing the percolation threshold and critical amplitude. In this paper we shall show that an analogous family of decorations exists for site problems, and verify that these families share the same exponents α, β, γ, δ and ν.

In § 2 we introduce site decorations and discuss the effect of such decorations on the exponent β. In § 3 we establish conditions such that site decorations leave the exponents α, β, γ, δ and ν unchanged, and discuss various 'universal' constants of lattice-lattice scaling. In § 4 we extend the above results to a continuum system.

2. Site decorations

We define a complete *n-pole* to be a complete graph on n vertices and we say that two complete n-poles are connected by a *transmission line* if the two n-poles form K_{2n}, the complete graph on $2n$ vertices. We shall call edges joining vertices within a given pole *pole bonds*, and use the term *transmission bonds* for edges joining vertices on different poles. We shall decorate a lattice by replacing sites by complete n-poles and bonds by transmission lines. A one-dimensional example is illustrated in figure 1.

If L is a particular lattice, and $L^{\bar{n}}$ is the n-pole decoration, we consider the following surjection of configurations on $L^{\bar{n}}$ onto those of L. We say that a vertex v_j of L is occupied if there is at least one occupied vertex in p_j where p_j is the n-pole corresponding to the vertex v_j.

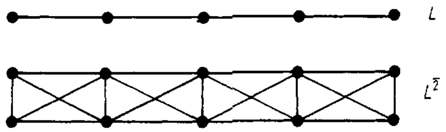

Figure 1. Example of site decoration in one dimension.

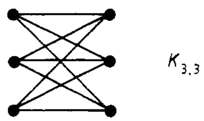

Figure 2. The complete bigraph $K_{3,3}$.

We notice that with this correspondence two adjacent poles in $L^{\bar{n}}$ (i.e. two poles directly connected by a transmission line) are connected if and only if the corresponding sites in L are connected. Furthermore configurations of randomly and independently distributed sites on $L^{\bar{n}}$ correspond to random independent configurations on L. The site density relation for the projection is

$$f(p)=1-(1-p)^n. \tag{2.1}$$

That is, a site density of p on $L^{\bar{n}}$ projects onto a density of $f(p)$ on L. The above observations allow us to state the following result.

Lemma 1. Let L be a lattice with site percolation exponent β and critical site density $p_c \in (0, 1)$. If $L^{\bar{n}}$ is a complete n-pole decoration of L then the percolation exponent for $L^{\bar{n}}$ is β and the critical density is the root of the equation

$$1-(1-d_c)^n-p_c=0, \qquad d_c \in (0,1).$$

The reasoning behind the proof of lemma 1 is similar to that of the bond decoration case discussed in oww. Namely $P(p)$, the percolation probability, is found to be related on the two lattices by the equation

$$P^{\bar{n}}(p)=c(p)P(f(p)), \qquad p>p_c$$

where $c(p)$ and $f(p)$ are analytic and non-zero in $(0, 1)$. This guarantees preservation of the exponent β provided $\mathrm{d}f/\mathrm{d}p|_{d_c}>0$ when d_c is the root of $f(d_c)-p_c=0$. (The proofs of exponent invariance under site decorations are in general lengthy but straightforward. We shall sketch the arguments in most cases; further details may be found in Ord (1983).)

We notice that if L is the covering lattice for a planar lattice L_β, then $L^{\bar{n}}$ is the covering lattice for $L_\beta^{\bar{n}}$ where $L_\beta^{\bar{n}}$ is the lattice L_β with every bond replaced by n bonds in parallel. If we reduce the connectivity of the poles then the covering property of the lattice is destroyed and the problem no longer has a bond equivalent. However, we shall see that this does not affect the percolation exponents.

An interesting feature of n-pole decorations is the fact that in clusters which span more than a single n-pole, by which we mean clusters which include sites of more than one n-pole, the pole bonds are redundant. This fact is illustrated in the following lemma.

Lemma 2. Let w be a collection of vertices in $L^{\bar{n}}$ with a single component section graph G and pole bond set B^{p}. If w contains vertices from more than a single pole then B^{p} does not contain a cut set of G.

This may be proved by noticing that any two vertices on a pole are connected by a path to an adjacent pole provided that the section graph G contains a vertex of that pole.

The above lemma suggests that we may modify the pole bond structure of an n-pole decoration without affecting its critical exponent. For example we may form a *minimal* n-pole decoration as follows. We call an n component graph of n vertices a minimal n-pole. Two minimal n-poles $\hat{p}_i$ and $\hat{p}_j$ are said to be connected by a transmission line in a graph G if the section graph induced in G by the vertices of $\hat{p}_i$ and $\hat{p}_j$ is the complete bigraph $K_{n,n}$ (figure 2). We notice that minimal n-poles have the same projected density function (2.1) as complete n-poles. Furthermore, we notice that if $L^{\underline{n}}$ is the decoration of L by minimal n-poles, then there is a one-to-one correspondence between configurations on $L^{\bar{n}}$ and $L^{\underline{n}}$. Thus any configuration on $L^{\underline{n}}$ may be obtained from the same configuration of sites on $L^{\bar{n}}$ by the removal of all pole bonds from $L^{\bar{n}}$. However, by lemma 2, removal of pole bonds from a configuration on $L^{\bar{n}}$ cannot affect clusters of size $N > n$. Thus the percolation probabilities on the two lattices at the same site density are identical, and lemma 1 also holds for minimal n-pole decorations.

The lattices $L^{\bar{n}}$ and $L^{\underline{n}}$ represent 'extremes' of n-pole decoration in the following sense. Let L^{n} be a decoration of $L^{\underline{n}}$ by the addition of *any* set of pole bonds (excluding double bonds) then $L^{\underline{n}} \subseteq L^{n} \subseteq L^{\bar{n}}$, and by the containment theorem (Fisher 1961) the percolation probability on L^{n} is bounded above and below by the percolation probabilities on $L^{\underline{n}}$ and $L^{\bar{n}}$, respectively. Since these are identical the percolation probability on L^{n} is identical to the percolation probability on $L^{\underline{n}}$ and so lemma 1 holds for any n-pole decoration.

The above irrelevance of pole bonds to percolation is a good example of the fact that critical behaviour tends to depend not on 'local' connectivity of a lattice, but on more 'global' structure. In fact, one may generalise n-pole decoration further to 'patch' decoration which may have considerable local structure that is essentially irrelevant to critical behaviour. That is, we define an n-patch to be a finite n-rooted graph (not necessarily connected). We call the bonds of an n-patch *patch bonds.* We form an n-patch decoration of L in the following way. We replace each vertex of L by an n-patch, and each bond in L by a transmission line joining the $2n$ roots of neighbouring n-patches (figure 3).

Since only root points of n-patches are directly connected to neighbouring patches, patch bonds are clearly irrelevant to the percolation probability of root vertices. Thus the actual patch structure affects the percolation probability only through the probability that a randomly chosen site is connected to an occupied root vertex. This probability is the analogue of the association probability in bond decoration (see oww). For a finite patch this association probability $h(p)$ is a polynomial and, as in the bond percolation case, only affects the amplitude of the singularity.

Finally, we may generalise this further to a stochastic patch decoration where the number of root vertices in a given patch is governed by independently chosen random variables from some distribution function $g(n)$, $n = 1, 2, 3, \ldots$. In this case the transmission line between an n-pole and an m-pole is just the complete bigraph $K_{m,n}$. As in the case of stochastic bond expansions, stochastic pole decoration, excluding 0-poles and infinite poles, does not change the exponent β (cf Ord and Whittington 1982).

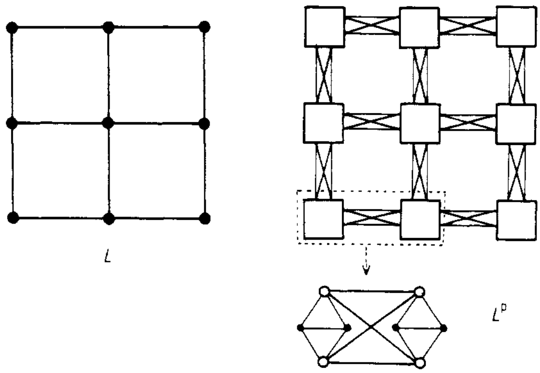

Figure 3. An n-patch decoration in two dimensions.

3. Decorations and exponent relations

In this section we shall consider the effect of site decorations on the exponents α, β, γ, δ and ν using series expansion techniques. We shall specifically consider a complete n-pole example where the pole bonds and sites form complete graphs on n vertices. The results may be readily extended to more locally complex decorations provided that the decorations remain finite.

Let us consider percolation on a lattice L where we expect to find critical behaviour. The cluster numbers on the lattice are defined by:

$n_s(p)$ = expected number of clusters of size s on the lattice (per site) $= p^s D_s(q)$

where $D_s(q)$ is the so-called 'perimeter' polynomial (see e.g., Stauffer 1979) which summarises the configurational data for s-clusters on L. The total number of clusters per occupied site on the lattice is then

$$K(p)=\frac{1}{p}\sum_s n_s(p)=\sum_s p^{s-1}D_s(q). \tag{3.1}$$

Let us assume that

$$K(p)\sim A|p-p_c|^{2-\alpha} \tag{3.2}$$

describes the behaviour of $K(p)$ in the vicinity of the critical density.

Similarly we write

$$\begin{aligned} P(p)&=1-\frac{1}{p}\sum_s sn_s(p)=1-R(p)\sim B|p-p_c|^{\beta} \qquad p\to p_c^+, \qquad \beta\geqslant 0\\ S(p)&=p^{-1}\sum_s s^2 n_s(p)\sim C|p-p_c|^{-\gamma}, \qquad \gamma\geqslant 0 \end{aligned} \tag{3.3}$$

and

$$M(p_c,h)=\frac{1}{p_c}\sum_s n_s(p_c)s\,e^{-hs}\sim M_0^{-1/\delta}h^{1/\delta}, \qquad \delta\geqslant 0$$

which define the exponents β, γ and δ.

To define the exponent ν we choose an arbitrary origin ($\boldsymbol{o}$) on L and consider the indicator function

$$\eta_L(\boldsymbol{r}) = \begin{cases} 1 & \text{if } \boldsymbol{r} \text{ is connected to } \boldsymbol{o} \text{ given that } \boldsymbol{o} \text{ is occupied} \\ 0 & \text{otherwise.} \end{cases}$$

The pair-connectedness function is then

$$C_L(\boldsymbol{r}, p) = \langle \eta_L(\boldsymbol{r}) \rangle p_{\mathrm{F}}$$

where the expectation is over all configurations of density p containing the origin and p_{F} is the fraction of sites in finite clusters. The mean-square displacement of sites connected to the origin is then

$$\xi^2(p) = \sum_{\boldsymbol{r}} r^2 C(\boldsymbol{r}, p) \Big/ \sum_{\boldsymbol{r}} C(\boldsymbol{r}, p)$$

and we assume

$$\xi(p) \sim \xi_0 |p - p_{\mathrm{c}}|^{-\nu}, \qquad \nu \geqslant 0. \tag{3.4}$$

In order to investigate the relationship between these quantities on L and the analogous quantities on $L^{\bar{n}}$ we consider an arbitrary s-cluster on $L^{\bar{n}}$. Any such cluster corresponds to a cluster of unique size $j \leqslant s$ on L. We may then write the perimeter polynomial of s-clusters on $L^{\bar{n}}$ as a sum of terms arising from the different possible contributions from L, i.e.,

$$\tilde{D}_s(q) = \sum_{j \geqslant s/n}^{s} \tilde{D}_s^j(q).$$

Here $\tilde{D}_s^j(q)$ is the contribution to $\tilde{D}_s(q)$ due to s-clusters that project onto j-clusters on L. However, for j-clusters on L we have:

$$D_j(q) = \sum_t g_{jt} q^t \tag{3.5}$$

where g_{jt} is the number of animals of size j and perimeter t on L. g_{jt} also represents the number of 'pole animals' on $L^{\bar{n}}$ with j poles and t perimeter poles. On $L^{\bar{n}}$ each perimeter pole has a weight of q^n. Furthermore each 'occupied' pole contributes configurationally due to its internal perimeter sites, i.e.,

$$\tilde{D}_s^j(q) = \frac{1}{n} \sum_t C_j^s g_{jt} q^{nt} q^{jn-s} \tag{3.6}$$

where q^{jn-s} represents the internal perimeter sites in 'occupied' poles and C_j^s is the number of ways of distributing s sites among j poles so that no pole is empty. The factor $1/n$ accounts for normalisation per site (as opposed to per pole). The generating function of the C_j^s is just

$$G^j(x) = \left(nx + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n \right)^j = \sum_{k=j}^{nj} C_j^k x^k. \tag{3.7}$$

Comparing (3.6) and (3.5) we may write

$$\tilde{D}_s^j(q) = (C_j^s/n) q^{jn-s} D_j(q^n). \tag{3.8}$$

Thus using (3.1) we may expand $\tilde{K}(p)$ for the lattice $L^{\bar{n}}$ as

$$
\begin{aligned}
\tilde{K}(p) = p^{-1}\sum_s \tilde{n}_s(p) = \sum_s \tilde{D}_s(q)p^{s-1} \\
&= \sum_s p^{s-1} \sum_{j\geq s/n}^{s} \tilde{D}_s^j(q) \\
&= \frac{1}{pn}\sum_s p^s \sum_{j\geq s/n}^{s} C_j^s q^{jn-s} D_j(q^n) \\
&= \frac{1}{pn}\sum_{j=1} D_j(q^n) \sum_{s=j}^{nj} C_j^s p^s q^{jn-s} \\
&= \frac{1}{pn}\sum_{j=1} D_j(q^n) q^{jn} G^j(p/q).
\end{aligned}
\tag{3.9}
$$

However

$$
\begin{aligned}
q^{jn}G^j(p/q) &= \left(npq^{n-1} + \binom{n}{2}p^2q^{n-2} + \ldots + \binom{n}{n}p^nq^0\right)^j \\
&= (1-q^n)^j.
\end{aligned}
\tag{3.10}
$$

Thus

$$
\begin{aligned}
\tilde{K}(p) &= \frac{1}{pn}\sum_{j=1} D_j(q^n)(1-q^n)^j \\
&= \frac{f(p)}{pn} K(f(p))
\end{aligned}
\tag{3.11}
$$

where $f(p) = (1-q^n)$.

In a similar way one can show that

$$
\begin{aligned}
\tilde{R}(p) &= R(f(p)) \\
\tilde{M}(p_c, h) &= \frac{(1-h)(1-(n-1)ph)}{(1-nph/f(p))} M(f(p), nph/f(p)) \qquad h \ll 1 \\
\tilde{S}(p) &= \frac{np}{f(p)} S(f(p)) + O(p)
\end{aligned}
\tag{3.12}
$$

where the tildes denote quantities evaluated on $L^{\bar{n}}$. Using slightly different techniques (see appendix) one can also show that

$$
\tilde{\xi}(p) = \xi(f(p)) + O(\tilde{S}^{-1}(p)). \tag{3.13}
$$

In OWW it was shown that a sufficient condition for the preservation of exponents under functional composition was that $f(p)$ be analytic and strictly increasing at p^* where p^* is the root of the equation $f(p^*) - p_c = 0$. These conditions are clearly satisfied for any $p_c \in (0, 1)$ and we conclude that the decorated lattice quantities of equations (3.11), (3.12) and (3.13) give rise to the same exponents $\alpha \ldots \nu$ as do the respective functions on L.

To investigate the amplitudes of the singularities we expand (3.11) as follows; denoting site density on $L^{\bar{n}}$ by d we have

$$\tilde{K}(d)=\frac{f(d)}{nd}K(f(d))$$

and

$$(1-d_c)^n=1-p_c.$$

Writing

$$p=f(d)=p_c+f'(d_c)(d-d_c)+f''(d_c)\tfrac{1}{2}(p-p_c)^2+\dots$$

and

$$K(p)\sim A|p-p_c|^{2-\alpha}$$

we have

$$\begin{aligned}\tilde{K}(d)&\sim(Af(d)/nd)|f'(d_c)(d-d_c)|^{2-\alpha}\\&\sim(Af(d_c)/nd_c)|f'(d_c)|^{2-\alpha}|d-d_c|^{2-\alpha}\end{aligned}$$

so that on the decorated lattice the critical amplitude is:

$$\tilde{A}=\frac{f(d_c)}{nd_c}(f'(d_c))^{2-\alpha}A. \tag{3.14}$$

In a similar manner we may calculate the other amplitudes. The results may be summarised as follows.

Lemma 3. Let L be a d-dimensional lattice with percolation threshold p_c and critical exponents $\alpha,\dots,\nu$ as defined in (3.2)-(3.6). The percolation threshold on $L^{\bar{n}}$ is then the root $d_c\in[0,1]$ of the equation $1-(1-d_c)^n-p_c=0$. Furthermore if $p_c\in(0,1)$ then the exponents of $L^{\bar{n}}$ are identical to those of L and the critical amplitudes are given by

$$\begin{aligned}&\tilde{A}=\frac{f(d_c)}{nd_c}f'(d_c)^{2-\alpha}A &&\tilde{B}=f'(d_c)^{\beta}B\\&\tilde{C}=\frac{nd_c}{f(d_c)}f'(d_c)^{-\gamma}C &&\tilde{M}^{-1/\delta}=\left(\frac{nd_c}{f(d_c)}\right)^{1/\delta}M^{-1/\delta}\\&\tilde{\xi}_0=f'(d_c)^{-\nu}\left(\frac{nd_c}{f(d_c)}\right)^{1/d}\xi_0.\end{aligned} \tag{3.15}$$

The relationships between critical amplitudes on the decorated lattices given above allow us to check the universal amplitude ratios. These amplitude ratios are usually derived from the hypotheses of scaling and universality (see Aharony 1980). However, we shall find that in the case of site decorated lattice families we need only certain exponent equalities for the ratios to be universal within families. The ratios in question are:

$$\begin{aligned}&R_x=C^+MB^{\delta-1}\\&R_c=\alpha(2-\alpha)(1-\alpha)A^+B^{-2}C^+\end{aligned} \tag{3.16}$$

and

$$(R_\xi^+)^d=\alpha(1-\alpha)(2-\alpha)A^+(\xi_0^+)^d.$$

For example, on the decorated lattice we have from (3.15)

$$\tilde{R}_x = \frac{nd_c}{f(d_c)} f'(d_c)^{-\gamma} \left(\frac{nd_c}{f(d_c)} \right)^{-1} f'(d_c)^{\beta(d-1)} CMB^{\delta-1} = f'(d_c)^{\beta(\delta-1)-\gamma} R_x.$$

Since in general $f'(d_c) \neq 1$ the amplitude ratio R_x is universal within a lattice family provided $\gamma = \beta(\delta-1)$. Similarly one has

$$\tilde{R}_c = f'(d_c)^{2-\alpha-2\beta-\gamma} R_c$$

or R_c is universal provided $2-\alpha = \gamma + 2\beta$, and

$$\tilde{R}_\xi = f'(d_c)^{2-\alpha-d\nu} R_\xi$$

or R_ξ is universal provided $d\nu = 2-\alpha$.

In six dimensions the last of equations (3.16) is replaced by (Aharony 1980)

$$K(p)(\xi(p))^6 \simeq D_0 |\ln|(p-p_c)/p_c t_0|| \tag{3.17}$$

with D_0 being universal and t_0 being non-universal. However, both relations (3.16) and (3.17) are satisfied by lattice families which suggest that t_0 is constant within families.

In cases where $p_c = 1$ (e.g., $1-d$ percolation) lemma 3 is no longer useful; however, a weaker form may be stated using the renormalised exponents of Suzuki (1974), namely

$$\hat{\beta} = \beta/\nu \qquad \hat{\gamma} = \gamma/\nu \qquad (2-\hat{\alpha}) = (2-\alpha)/\nu \qquad \text{and} \qquad \hat{\delta} = \delta.$$

Corollary. If the lattice L is such that $p_c = 1$ and scaling relations (3.11)-(3.13) hold, then the decorated lattices share identical renormalised exponents.

This result may be inferred from relations (3.11) and (3.12) by expanding $f(p)$ about $p=1$. The vanishing of the first $(n-1)$ derivatives of $f(p)$ at $p=1$ increases the exponents β, γ, $(2-\alpha)$ and ν by a factor of n while leaving δ unchanged. The renormalised exponents are thus unchanged by the decoration.

It is interesting to note that the weaker universality of the corollary applies directly to one-dimensional percolation, whereas all common higher-dimensional lattices obey the stronger universality of lemma 3.

4. A continuum limit

The above lemma suggests that, at least with respect to site decorations, critical exponents are universal. In fact, there appear to be only two possible ways to change exponents with site decorations. From the above lemma if $p_c = 1$ then $d_c = 1$ and $f'(d_c) = 0$. This reduces the critical amplitude of the decorated lattices to zero and leads to a change of critical exponents by integer multiples (as found in some tangential approaches to critical lines, see e.g., Griffiths and Wheeler (1970)).

Another possibility is to allow infinite decorations. Although we shall show that we may allow the n-pole decoration to become infinite in such a way as to preserve the exponents, this is not the general case for infinite patch decorations.

Imagine a sequence of decorations of the quadratic lattice with first and second neighbour interactions, denoted $S^{\sqrt{2}}$. We structure the sequence of decorations so that the vertex set of each member can be arranged in such a way as to be a refinement of the previous member (figure 4). That is, suppose we start with a nearest neighbour distance $b=1$ on $S^{\sqrt{2}}$ and at the nth member of the sequence we have $b(n)=(\frac{1}{3})^n$. This will require the number of sites per decoration to be $N(n)=3^{2n}$. If S^x is the quadratic lattice with all connections up to and including x lattice constants present, and L^k is $S^{\sqrt{2}}$ decorated with 3^{2k}-poles as above we have

$$S^{3^n} \subset L^n \subset S^{2\sqrt{2}\,3^n}$$

where the containment symbols indicate identity in vertex sets and strict containment in bond sets. (To compare lattices we make the nearest-neighbour distance on S^{3^n} and $S^{2\sqrt{2}\,3^n}$, $b(n)=(\frac{1}{3})^n$.) As $n\to\infty$ percolation on S^{3^n} and $S^{2\sqrt{2}\,3^n}$ approach continuum percolation of overlapping discs of radii 1 and $2\sqrt{2}$, respectively. Although it is not known whether percolation exponents change along these two sequences, particularly in the limit $n\to\infty$, by lemma 3 we know that the sequence $\{L^n\}$ has identical exponents.

To compare members of $[L^n]$ we denote site density per unit area by ρ. If d is the density on L^n then $\rho = d3^{2n}$.

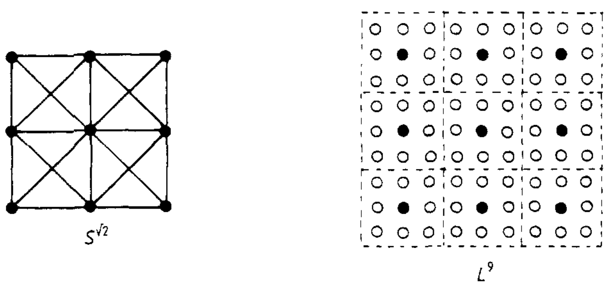

Figure 4. $S^{\sqrt{2}}$ is a square lattice with first and second-neighbour bonds. L^9 is a pole decoration of this lattice (only sites are shown) such that the sites of the decorated lattice form a square array in the plane.

Consider the sequence of functions defined by:

$$P_n(\rho) \equiv P^{(n)}(d=\rho/3^{2n}) = P^{(0)}[1-(1-\rho/3^{2n})^{3^{2n}}]$$

where we have $n \geqslant \ln\rho/\ln 3$, and $P^{(n)}$ is the percolation probability on the complete 3^{2n} decoration of $S^{\sqrt{2}}$.

If we assume $P^{(0)}(\rho)$ continuous on [0, 1] then we have for fixed ρ

$$\begin{aligned}\lim_{n\to\infty} P_n(\rho) &= \lim_{n\to\infty} P^{(0)}[1-(1-\rho/3^{2n})^{3^{2n}}]\\ &= P^{(0)}(1-\mathrm{e}^{-\rho})\\ &\equiv P(\rho)\end{aligned}$$

where $P(\rho)$ is the limiting percolation probability. Since $P_n(\rho)$ converges to $P^{(0)}(1-\mathrm{e}^{-\rho})$ pointwise on [0, 1] and since $P^{(0)}$ is assumed continuous we need only establish that the function $F(\rho)=1-\mathrm{e}^{-\rho}$ is such that $F(\rho_c)=d_c$ has a unique solution, where d_c is the percolation threshold on $L^{(0)}$, and that $F'(\rho_c)\neq 0$. This will show that the exponent β for the continuum is the same as for the lattice. However, $\mathrm{e}^{-\rho}$ is strictly decreasing on $[0, +\infty)$ so that $F(\rho)$ is strictly increasing on that interval. Thus for $d_c \in [0, 1)$ the solution is unique and $F'(\rho_c) \neq 0$.

We thus see that the derived continuum problem with percolation probability $P(\rho)$ has a percolation exponent (β) identical to that of the parent lattice. We may also use the arguments of § 2 to confirm the preservation of α, γ, δ and ν into the continuum. Furthermore, we may extend the arguments to higher dimensions and verify the preservation of the constants R_x, R_c and R_ξ.

The above sequence of lattices can be regarded as approximations to the limiting case of cell percolation on a Bravais lattice. By cell percolation we mean the following. Consider a Bravais lattice β, and partition space into Wigner-Seitz cells about the lattice points. Consider two cells to be adjacent in a percolation sense if the enclosed sites of the lattice β share a common bond. Now consider a random independent distribution of points in the space partitioned by the Wigner-Seitz cells of β. Connect those points that are in adjacent cells. This cell percolation is the limiting case of the pole decorations mentioned above. We note that cell percolation allows for multiple occupancy and random placement within cells, and that the critical behaviour does not depend on any assumed connectivity within cells. We call a decoration limit problem of a Bravais lattice B a cell-equivalent problem, and the above considerations lead to the following result.

Lemma 4. Let G be a cell-equivalent problem for a Bravais lattice B with critical density $p_c \in (0, 1)$. The critical density on G is then $p_c = -\ln(1-p_c)$ and the exponents $\alpha \ldots \nu$ on G are identical to those on B, with critical amplitudes given by equations (3.15) with $f(p) = 1 - e^{-p}$.

5. Discussion

We have shown that one may use site decorations to construct lattice families, with predictably varying percolation thresholds and critical amplitudes, in which the exponents remain unchanged. These families include planar, non-planar, covering and non-covering lattices as well as cell percolation, which allows for multiple occupancy and random placement within cells. We have also established that the universal amplitude ratios are constant within families provided only that the corresponding exponent relations hold.

Appendix

In the following we consider percolation on a d-dimensional hypercubic lattice L with nearest-neighbour distance $b_0 = 1$. We shall compare this to percolation on a complete n-pole decorated lattice $L^{\bar{n}}$. For convenience we consider values of n such that $n = (2n_0+1)^d$ for some integer $n_0 \geq 1$. For such n we may choose a nearest-neighbour distance $b_n = (2n_0+1)^{-1}$ on L^n. We may then regard $L^{\bar{n}}$ as a refinement of L in which one in every $2n_0+1$ vertices in L^n corresponds to a vertex in L (figure 4).

We choose an arbitrary origin in L and denote the set of vertex coordinates by

$$V = \{\boldsymbol{R} = (l_1, l_2, \ldots l_d) | l_1, \ldots l_d \in Z\}.$$

For brevity we identify a vertex in L by its position vector $\boldsymbol{R}$, and a vertex in $L^{\bar{n}}$ by an ordered pair $(\boldsymbol{R}, \boldsymbol{S})$ where the vector $\boldsymbol{S}$ ranges over the set

$$S_0 = \left\{ \boldsymbol{S} = \frac{1}{2n_0+1}(l_1, \ldots l_d) \middle| -n_0 \leq l_1, \ldots l_d \leq n_0, l_1, \ldots l_d \in Z \right\}$$

and where the position of $(\boldsymbol{R}, \boldsymbol{S})$ is just $\boldsymbol{R}+\boldsymbol{S}$.

Denote by F the event that the origin is occupied and contained in a finite cluster of occupied sites. We then define the indicator function:

$$\eta_{L^{\bar{n}}}(\boldsymbol{R}, \boldsymbol{S}) = \begin{cases} 1 & \text{if } (R, S) \text{ is connected to the origin in } L^{\bar{n}} \text{ given } F \\ 0 & \text{otherwise.} \end{cases}$$

For any given configuration on $L^{\bar{n}}$ we consider the configuration on L obtained by occupying only those positions $\boldsymbol{R}$ for which there is at least one occupied vertex in the pole $(\boldsymbol{R}, \boldsymbol{S})$ in $L^{\bar{n}}$. Random, independently distributed configurations of occupied sites at density d on $L^{\bar{n}}$ generate random independently distributed configurations on L with site density $p = 1-(1-d)^n$. The indicator function for L can thus be written

$$\eta_L(\boldsymbol{R}) = \begin{cases} 1 & \text{if} \sum_{\boldsymbol{S}} \eta_{L^{\bar{n}}}(\boldsymbol{R}, \boldsymbol{S}) > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{A1}$$

Since all vertices on a given complete n-pole are equivalent with respect to all neighbouring sites we have

$$\langle \eta_{L^{\bar{n}}}(\boldsymbol{R}, \boldsymbol{S}_1)\rangle = \langle \eta_{L^{\bar{n}}}(\boldsymbol{R}, \boldsymbol{S}_j)\rangle \qquad j = 1, \ldots, n. \tag{A2}$$

Furthermore, from 1 we have $\eta_{L^{\bar{n}}}(\boldsymbol{R}, \boldsymbol{S}) = 1$ iff $\eta_L(\boldsymbol{R}) = 1$ and both $(\boldsymbol{R}, \boldsymbol{S})$ and $(\boldsymbol{0}, \boldsymbol{0})$ are occupied, so that

$$\begin{aligned} \langle \eta_{L^{\bar{n}}}(\boldsymbol{R}, \boldsymbol{S})\rangle &= \langle \eta_L(\boldsymbol{R})\rangle \times \Pr\{(\boldsymbol{R}, \boldsymbol{S}) \text{ and } (\boldsymbol{0}, \boldsymbol{0}) \text{ occupied} | \eta_L(\boldsymbol{R}) = \eta_L(\boldsymbol{0}) = 1\} \\ &= \langle \eta_L(\boldsymbol{R})\rangle d_p^2 \end{aligned} \tag{A3}$$

where the left-hand side is evaluated at site density d and the right-hand side is evaluated at site density $p = 1-(1-d)^n$ with $d_p = d/p$.

The expected size of the cluster at the origin, given F, is then

$$\begin{aligned} S_{L^{\bar{n}}}(d) &= \sum_{\boldsymbol{R}} \sum_{\boldsymbol{S}} \langle \eta_{L^{\bar{n}}}(R, S)\rangle \\ &= nd_p^2 \sum_{\boldsymbol{R}} \langle \eta_L(\boldsymbol{R})\rangle \\ &= nd_p^2 S_L(p). \end{aligned}$$

Thus

$$\begin{aligned} \xi^2_{L^{\bar{n}}}(d) &= [nd_p^2 S_L(p)]^{-1} \sum_{\boldsymbol{R}} \sum_{\boldsymbol{S}} (\boldsymbol{R}+\boldsymbol{S})^2 \langle \eta_{L^{\bar{n}}}(\boldsymbol{R}, \boldsymbol{S})\rangle \\ &= [nd_p^2 S_L(p)]^{-1} \left(\sum_{\boldsymbol{R}} nd_p^2 R^2 \langle \eta_L(R)\rangle + \sum_{\boldsymbol{R}} \sum_{\boldsymbol{S}} (S^2 + 2RS \cos \theta_{RS}) \langle \eta_L(R)\rangle d_p^2 \right) \\ &= \xi_L^2(p) + [nS_L(p)]^{-1} \sum_{\boldsymbol{R}} \sum_{\boldsymbol{S}} (S^2 + 2RS \cos \theta_{RS}) \langle \eta_L(R)\rangle \end{aligned}$$

where θ_{RS} is the angle between $\boldsymbol{R}$ and $\boldsymbol{S}$.

The first term in the sum on the right-hand side is bounded by the largest value of S^2, namely: $S^2_{\max} = [dn_0^2/(2n_0+1)]^{2/d}$. The second term, when summed, is zero by symmetry. (For other lattices in which this is not the case one may easily show that the term, while non-zero, is less singular than $\xi^2(p)$ for $\nu > 0$.)

We thus have

$$\xi^2_{L^n}(d) = \xi^2_L(p) + O(1)$$

and since the two correlation lengths are related by functional composition of a smooth increasing function, they share the same exponent ν provided $p_c \in (0, 1)$.

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